Theory of quantum fluctuations in classically chaotic Hamiltonian systems

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In a number of numerical experiments it has been demonstrated that the initial growth of quantum variances of the dynamical variables for a chaotic trajectory is exponential in nature. This is a typical signature of classical chaos on a generic quantum dynamical feature. Based on the theory of multiplicative noise we have proposed a quantitative theory of this exponential divergence of quantum dispersions for general Hamiltonian systems, the rate constant being determined by the correlation function of the fluctuations of the curvature of the classical potential. The theory has been subsequently applied to a model driven double-well oscillator with detailed classical and quantum-mechanical calculation to verify the theoretical propositions. [S1063-651X(96)10008-8]

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I. INTRODUCTION

Chaos in dynamical systems is one of the key issues in nonlinear physics today [1,2]. The chaotic motion is not associated with the variation of stochastic parameters or force, but is intrinsically due to unstable character of trajectories in phase space. To be more specific, the instability is because of exponential separation of initially nearby trajectories. The rate of growth of the separation is measured by the largest Lyapunov exponent. Very recently we have proposed [3] a general relationship between fluctuations and diffusion for Hamiltonian systems which relates the largest Lyapunov exponent to the fluctuations of the curvature of the potential in a way that is reminiscent of the Kubo relations in statistical physics, so that the exponent can be viewed as a transport coefficient in phase space. We have also shown [4] that the theory of multiplicative noise can be a good natural description of classical chaos in several occasions.

Recently a number of numerical experiments have demonstrated [5–7] that the initial growth of quantum variances of dynamical variables, such as position or momentum for a classical trajectory, is exponential in nature. This is a typical signature of classical chaos on a generic dynamical feature. The object of this paper is to propose a theory for this exponential growth of quantum variances. We have shown that correlation between fluctuations in the curvature of the classical potential which is amenable to a stochastic description in terms of the theory of multiplicative noise [8] also determines the rate of initial growth of dispersion. Based on Wigner quantum-classical correspondence [9,10] we have derived appropriate Fokker-Planck equations where the drift and diffusion terms have their origin in dynamical properties of fluctuations of the curvature of the classical potential. Using detailed classical and quantum-mechanical calculations, the semiclassical analysis of the dynamics of quantum variances has been numerically examined and verified in a model driven double-well oscillator which admits classical chaos.

The organization of the paper is as follows: In Sec. II we briefly present the $\hbar$ scaling of the Wigner equation approach of Fox and Elston [10], followed by a detailed treatment of classical fluctuations of the curvature of the potential using the theory of multiplicative noise in Sec. III. An application of the master equation for the quantum fluctuation distribution function in the case of a driven double-well oscillator is given in Sec. IV. The paper is concluded in Sec. V.

II. $\hbar$ SCALING OF WIGNER EQUATION

To start with we consider the quasiclassical distribution approach of Wigner. Over the years this has proved to be a standard starting point of analysis of quantum-classical correspondence. The Wigner distribution function is defined in phase space $\{x_i, p_i\}$ as follows:

$$W(\{x_i\},\{p_i\}) = [1/(\hbar\pi)^N] \int \cdots \int d\xi_1 \cdots d\xi_N \psi^*(\{x_i+\xi_i\}) \times \psi(\{x_i-\xi_i\}) \exp\left[(2i/\hbar) \left(\sum_i p_i \xi_i \right) \right], \quad (1)$$

where $\psi(x)$ refers to the quantum wave function of an N-degree-of-freedom Hamiltonian system.

The time evolution of the Wigner function W of a dynamical system characterized by a Hamiltonian of the form

$$H = \sum_{i=1}^{N} (p_i^2/2m_i) + V(\{x_i\}) \quad (2)$$

is given by

$$\frac{\partial W}{\partial t} = \sum_{i=1}^{N} \left[-\frac{p_i}{2m_i} \frac{\partial W}{\partial x_i} + \left(\frac{\partial V}{\partial x_i} \right) \frac{\partial W}{\partial p_i} \right] + \sum_{\substack{n_1+n_3+\cdots n_N \\ \text{odd}}} \left(\frac{\partial^{n_1+\cdots+n_N} V}{\partial x_1^{n_1} \cdots \partial x_N^{n_N}} \right) \frac{(\hbar/2i)^{n_1+\cdots+n_N-1}}{n_1! \ldots n_N!} \times \frac{\partial^{n_1+\cdots+n_N} W}{\partial p_1^{n_1} \cdots \partial p_N^{n_N}}. \quad (3)$$

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One also recovers the Liouville equation in the limit $\hbar \to 0$.

Following Ref. [10] it is now convenient to introduce the scaling

$$x_i = x_i(t) + \hbar^{1/2}\mu_i, \qquad p_i = p_i(t) + \hbar^{1/2}\nu_i, \tag{4}$$

where $\hbar$ is the smallness parameter analogous to the expansion parameter employed in Van Kampen's system size expansion. μ and ν in Eq. (4) refer to fluctuations in the coordinate and momentum, respectively. The time evolution of the fluctuation distribution function obeys

$$\frac{\partial \phi(\{\mu_i\},\{\nu_i\},t)}{\partial t} = \sum_k \left[-\frac{\nu_k}{m_k}\frac{\partial \phi}{\partial \mu_k} + \mu_j \frac{\partial^2 V}{\partial x_j \partial x_k}\frac{\partial \phi}{\partial \nu_k}\right] + O(\hbar^{1/2}). \tag{5}$$

The initial distribution satisfies

$$\langle (\Delta\mu_i)^2\rangle^{1/2}\langle (\Delta\nu_i)^2\rangle^{1/2} = \sigma\,\frac{1}{2\sigma} = \frac{1}{2}, \tag{6}$$

where σ specifies the spread in the initial distribution of fluctuations

$$\phi(\mu,\nu,0) = \prod_{k=1}^{N} \frac{1}{4\pi}\exp\left[-\frac{\mu_k^2}{2\sigma^2} - 2\sigma^2\nu_k^2\right]. \tag{7}$$

To put Eq. (5) in a more compact form, it is convenient to invoke the symplectic structure of Hamiltonian dynamics. For this, we specify

$$z_i = \begin{cases} x_i & \text{for } i = 1\ldots N \\ p_{i-N} & \text{for } i = N+1\ldots 2N. \end{cases}$$

Defining I as

$$I = \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix},$$

where E is an $N \times N$ unit matrix, the Hamilton equations become

$$\dot{z}_i = \sum_j I_{ij}\frac{\partial H}{\partial z_j}. \tag{8}$$

Again introducing scaling of z_i as

$$z_i = z_i(t) + \hbar^{1/2}\eta_i, \tag{9}$$

with

$$\eta_i = \mu_i \quad \text{for } i = 1\ldots N, \qquad = \nu_{i-N} \quad \text{for } i = N+1\ldots 2N,$$

one obtains the equation of motion for the fluctuation distribution function as

$$\frac{\partial \phi(\boldsymbol{\eta},t)}{\partial t} = -\sum_{i,j} J_{ji}\eta_i \frac{\partial \phi(\boldsymbol{\eta},t)}{\partial \eta_j}. \tag{10}$$

Here

$$J_{ij} = \sum_k I_{ik}\frac{\partial^2 H}{\partial z_k \partial z_j} \tag{11}$$

contains the second derivative of the potential. The matrix J is thus determined by the nature of the classical motion. For a chaotic trajectory, the fluctuations in J thus affect the evolution of quantum fluctuations. We address this aspect in Sec. III of the paper.

III. TREATMENT OF FLUCTUATIONS OF THE CURVATURE OF THE POTENTIAL

Equation (10) may be rewritten in a more compact form as follows:

$$\frac{\partial \phi}{\partial t} = -\mathbf{F}(t)\cdot\boldsymbol{\nabla}\phi, \tag{12}$$

where

$$\mathbf{F} = J\cdot\boldsymbol{\eta}, \tag{13}$$

and $\boldsymbol{\nabla}$ refers to differentiation with respect to components of $\boldsymbol{\eta}$.

Before proceeding further we would like to emphasize two relevant points at this stage. First, we consider a fully developed chaotic regime, i.e., the measure of a regular region is overwhelmingly small so that the dynamical variables $\{z_i\}$ may be treated as stochastic variables. Equations (11) and (13) then imply that $\mathbf{F}$ in Eq. (12), which incorporates the fluctuations of the curvature of the potential, is a stochastic process.

Second, in our theoretical and numerical considerations that follow we do not make any *a priori* assumption on the nature of stochastic process $\mathbf{F}(t)$. The special case that a stochastic process is a Gaussian or δ-correlated process, etc., have received so much attention in the recent literature that it is necessary to note that no such ad hoc assumption has been made. Equation (12) may therefore be interpreted as a stochastic differential equation with multiplicative noise.

The characteristic curves corresponding to Eq. (12) admit of general stochastic nonlinear differential equations

$$\dot{\eta}_i = F_i(\eta_1\ldots\eta_N,t), \quad i = 1\ldots N, \tag{14}$$

Since the nonlinearity in Eq. (14) generates higher moments, one cannot expect to find a differential equation for $\langle\boldsymbol{\eta}\rangle$. Thus one has to work with linear Eq. (12) directly.

We now rewrite Eq. (12) as

$$\dot{\phi} = [A_0 + \alpha A_1(t)]\phi, \tag{15}$$

where

$$A_0 = -\mathbf{F}_0\cdot\boldsymbol{\nabla}$$

and

$$A_1 = -\mathbf{F}_1\cdot\boldsymbol{\nabla}. \tag{16}$$

Here partitioning of $\mathbf{F}$ into $\mathbf{F}_0$ and $\alpha\mathbf{F}_1$ implies that $\mathbf{F}$ contains a constant part $\mathbf{F}_0$ and a fluctuating part $\mathbf{F}_1(t)$ which

gives rise to instability. Here α is a smallness parameter required to keep track of the strength of fluctuations due to classical chaos. The symbol ∇ is used for the operator that differentiates everything that comes after it with respect to the components of $\boldsymbol{\eta}$.

One of the main results for the linear equations of the form (15) with multiplicative noise [8] may now be in order. The average of ϕ obeys

$$\langle\dot{\phi}\rangle=\left\{A_0+\alpha\langle A_1\rangle+\alpha^2\int d\tau\langle\langle A_1(t)\exp[\tau A_0]\times A_1(t-\tau)\rangle\rangle\exp[-\tau A_0]\right\}\langle\phi\rangle. \tag{17}$$

The above result is based on the second-order cumulant expansion, and is valid in the case when fluctuations are small but rapid and correlation time τ_c is short but finite, or more precisely,

$$\langle\langle A_1(t)A_1(t')\rangle\rangle=0 \quad \text{for } |t-t'|>\tau_c. \tag{18}$$

We have, in general, $\langle A_1\rangle\neq 0$. Here $\langle\langle\ldots\rangle\rangle$ implies

$$\langle\langle q_iq_j\rangle\rangle=\langle q_iq_j\rangle-\langle q_i\rangle\langle q_j\rangle.$$

Equation (17) is exact in the limit correlation time τ_c tends to zero. Using relations (16), we obtain

$$\frac{\partial\langle\phi\rangle}{\partial t}=\left\{-\mathbf{F}_0\cdot\nabla-\alpha\langle\mathbf{F}_1(t)\cdot\nabla\rangle+\alpha^2\int_0^\alpha d\tau\langle\langle\mathbf{F}_1(t)\cdot\nabla\times\exp(-\tau\mathbf{F}_0\cdot\nabla)\mathbf{F}_1(t-\tau)\cdot\nabla\rangle\rangle\exp(\tau\mathbf{F}_0\cdot\nabla)\right\}\langle\phi\rangle. \tag{19}$$

The operator $\exp(-\tau\mathbf{F}_0\cdot\nabla)$ provides the solution of the equation

$$\frac{\partial f(\boldsymbol{\eta},t)}{\partial t}=-\mathbf{F}_0\cdot\nabla f(\boldsymbol{\eta},t) \tag{20}$$

(f signifies the ''unperturbed'' part of $\langle\phi\rangle$) which can be found explicitly in terms of characteristic curves. The equation

$$\boldsymbol{\eta}=F_0(\boldsymbol{\eta}) \tag{21}$$

for fixed t determines a mapping from $\eta(\tau=0)$ to $\eta(\tau)$, i.e., $\eta\rightarrow\eta^\tau$ with inverse $(\eta^\tau)^{-\tau}=\eta$. The solution of (20) is

$$f(\boldsymbol{\eta},t)=f(\boldsymbol{\eta}^{-t},0)\left|\frac{d(\boldsymbol{\eta}^{-t})}{d(\boldsymbol{\eta})}\right|=\exp[-t\mathbf{F}_0\cdot\nabla]f(\boldsymbol{\eta},0), \tag{22}$$

$|d(\boldsymbol{\eta}^{-t})/d(\boldsymbol{\eta})|$ being a Jacobian determinant.
The effect of $\exp(-t\mathbf{F}_0\cdot\nabla)$ on $f(\boldsymbol{\eta})$ is as follows:

$$\exp(-\mathbf{tF}_0\cdot\nabla)f(\boldsymbol{\eta},0)=f(\boldsymbol{\eta}^{-t},0)\left|\frac{d\boldsymbol{\eta}^{-t}}{d\boldsymbol{\eta}}\right|. \tag{23}$$

This simplification yields

$$\frac{\partial\langle\phi\rangle}{\partial t}=\left\{-\mathbf{F}_0\cdot\nabla-\alpha\langle\mathbf{F}_1(t)\cdot\nabla\rangle+\alpha^2\int_0^\alpha d\tau\left|\frac{d\boldsymbol{\eta}^{-\tau}}{d\boldsymbol{\eta}}\right|\times\langle\langle F_1(\boldsymbol{\eta},t)\cdot\nabla_\tau\mathbf{F}_1(\boldsymbol{\eta}^{-\tau},t-\tau)\rangle\rangle\nabla_{-\tau}\left|\frac{d\boldsymbol{\eta}}{d\boldsymbol{\eta}^{-\tau}}\right|\right\}\langle\phi\rangle. \tag{24}$$

Since Eq. (12) neglects the effect of higher powers of h, the above equation is a semiclassical equation for quantum fluctuation distribution function. At the same time the equation is second order in α, i.e., to order $\alpha^2\tau_e$ (with respect to the strength of classical fluctuations). Since Eq. (24) contains second-order derivatives with respect to the components of $\boldsymbol{\eta}$, it has the form of a Fokker-Planck equation.

IV. APPLICATION

A. Theoretical considerations

As an application we consider a model double-well oscillator characterized by its position x and momentum p and driven by a classical field of frequency ω_0. The Hamiltonian [11] is given by

$$H=p^2/2m+ax^4-bx^2+gx\ \cos\omega_0t. \tag{25}$$

Here the first term represents the kinetic energy, the second and third terms comprise the potential energy of the double-well oscillator, and the remaining part is the driving term. g includes the effect of coupling of the system with the field as well as the strength of the field. The classical equations of motion are

$$\begin{aligned}\dot{x}&=p,\\ \dot{p}&=-4ax^3+2bx-g\ \cos\omega_0t.\end{aligned} \tag{26}$$

The equations of motion for quantum fluctuation variables η_1 and η_2 corresponding to x and p [Eq. (14)] read as follows:

$$\frac{d}{dt}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}=J\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}, \tag{27}$$

where J is expressed as in earlier notation, and

$$z_1=x \quad \text{and} \quad z_2=p$$

are given by

$$J=\begin{bmatrix}0 & E\\ -E & 0\end{bmatrix}\begin{bmatrix}\dfrac{\partial^2H}{\partial z_1\partial z_1} & \dfrac{\partial^2H}{\partial z_1\partial z_2}\\ \dfrac{\partial^2H}{\partial z_2\partial z_1} & \dfrac{\partial^2H}{\partial z_2\partial z_2}\end{bmatrix}.$$

J is then reduced to

$$\begin{bmatrix}0 & \dfrac{1}{m}\\ \xi(t)+2b & 0\end{bmatrix},$$

where

$$\xi(t)=-12ax^2.$$

Therefore we have

$$\frac{d}{dt}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}=\mathbf{F}_0+\mathbf{F}_1, \tag{28}$$

with

$$\mathbf{F}_0=\begin{bmatrix}(1/m)\,\eta_2\\ 2b\,\eta_1\end{bmatrix} \quad \text{and} \quad \mathbf{F}_1=\begin{bmatrix}0\\ \xi(t)\,\eta_1\end{bmatrix},$$

where $\mathbf{F}_0$ and $\mathbf{F}_1$ are the constant and fluctuating parts, respectively. The fluctuations in F_1, i.e., in $\xi(t)$, are due to the stochasticity of the classical dynamical equation of motion (26). Now for the sure part we write

$$F_{0,1}=\frac{1}{m}\,\eta_2, \quad F_{0,2}=2b\,\eta_1,$$

and for the fluctuating part we have, similarly,

$$F_{1,1}=0, \quad F_{1,2}=\xi(t)\,\eta_1.$$

Without necessarily being δ correlated, $\zeta(t)$ has a short but finite autocorrelation time τ_c. We may now apply the result of Eq. (24).

The mapping $\eta\rightarrow\eta^t$ is found by solving the ''unperturbed'' equations

$$\dot{\eta}_1=\frac{1}{m}\,\eta_2,$$

$$\dot{\eta}_2=2b\,\eta_1.$$

As a short time approximation we consider the variation of η_1 and η_2 during τ_c,

$$\begin{aligned}\eta_1^{\tau}&=\frac{\tau}{m}\,\eta_2+\eta_1,\\ \eta_2^{\tau}&=m\lambda^2\tau\eta_1+\eta_2.\end{aligned} \tag{29}$$

where $\lambda=\sqrt{2b/m}$.

The Jacobian determinant [Eq. (24)] of this transformation or mapping reads

$$\left|\frac{d\boldsymbol{\eta}^{-\tau}}{d\boldsymbol{\eta}}\right|=\begin{bmatrix}1 & -\tau/m\\ -m\lambda^2\tau & 1\end{bmatrix}=1-\lambda^2\tau^2\approx 1,$$

and also we note

$$\left|\frac{d\boldsymbol{\eta}}{d\boldsymbol{\eta}_{-\tau}}\right|\approx 1. \tag{30a}$$

Next we consider the derivative terms in Eq. (24). The first term of the right hand side in Eq. (24) can be written as follows:

$$\mathbf{F}_0\cdot\boldsymbol{\nabla}=\sum_j F_{0j}\,\frac{\partial}{\partial\eta_j}=\frac{1}{m}\,\eta_2\,\frac{\partial}{\partial\eta_1}+2b\,\eta_1\,\frac{\partial}{\partial\eta_2}. \tag{30b}$$

Similarly we have

$$\mathbf{F}_1\cdot\boldsymbol{\nabla}=\sum_j F_{1j}\,\frac{\partial}{\partial\eta_j}=\xi(t)\,\eta_1\,\frac{\partial}{\partial\eta_2} \tag{30c}$$

and

$$\begin{aligned}\mathbf{F}_1(\boldsymbol{\eta}^{-\tau},t-\tau)\cdot\boldsymbol{\nabla}_{-\tau}&=\sum_j F_{1j}(t-\tau)\,\frac{\partial}{\partial\eta_j^{-\tau}}\\ &=\xi(t-\tau)\,\eta_1^{-\tau}\,\frac{\partial}{\partial\eta_2^{-\tau}}.\end{aligned}$$

The average and cumulants in Eq. (24) can therefore be expressed as

$$\langle\mathbf{F}_1(\boldsymbol{\eta},t)\cdot\boldsymbol{\nabla}\rangle=\langle\xi(t)\rangle\,\eta_1\,\frac{\partial}{\partial\eta_2}, \tag{30d}$$

$$\langle\langle\mathbf{F}_1(\boldsymbol{\eta},t)\cdot\boldsymbol{\nabla}_{\tau}\mathbf{F}_1(\boldsymbol{\eta}^{-\tau},t-\tau)\rangle\rangle\cdot\boldsymbol{\nabla}_{-\tau}$$

$$=\langle\langle\xi(t)\xi(t-\tau)\rangle\rangle\eta_1\,\frac{\partial}{\partial\eta_2}\,\eta_1^{-\tau}\,\frac{\partial}{\partial\eta_2^{-\tau}}.$$

Also note that differentiation with respect to $\eta_2^{-\tau}$ can be written in terms of η_1 and η_2 [see Eq. (29)];

$$\frac{\partial}{\partial\eta_2^{-\tau}}=\frac{\tau}{m}\,\frac{\partial}{\partial\eta_1}+\frac{\partial}{\partial\eta_2}$$

and

$$\eta_1^{-\tau}\,\frac{\partial}{\partial\eta_2^{-\tau}}=\frac{\tau}{m}\,\eta_1\,\frac{\partial}{\partial\eta_1}-\frac{\tau}{m}\,\eta_2\,\frac{\partial}{\partial\eta_2}+\eta_1\,\frac{\partial}{\partial\eta_2}. \tag{30e}$$

The above relation (30e) can be used to simplify (30d). Therefore we obtain

$$\langle\langle\mathbf{F}_1(\boldsymbol{\eta},t)\cdot\boldsymbol{\nabla}_{\tau}\mathbf{F}_1(\boldsymbol{\eta}_1^{-\tau},t-\tau\rangle\rangle\cdot\boldsymbol{\nabla}_{-\tau}$$

$$=\eta_1^2\,\frac{\partial}{\partial\eta_2^2}+\frac{\tau}{m}\,\eta_1^2\,\frac{\partial^2}{\partial\eta_2\partial\eta_1}-\frac{\tau}{m}\left(\eta_1\eta_2\,\frac{\partial^2}{\partial\eta_2^2}+\eta_1\,\frac{\partial}{\partial\eta_2}\right). \tag{31}$$

We are now in a position to write down the master equation (24) in the case of the model driven double-well system. This is

$$\begin{aligned}\frac{\partial\langle\phi\rangle}{\partial t}=&\left[\left\{-\frac{\eta_2}{m}\,\frac{\partial}{\partial\eta_1}-\left(2b+\alpha c+\alpha^2\,\frac{c_1}{m}\right)\eta_1\,\frac{\partial}{\partial\eta_2}\right\}\right.\\ &+\left\{\alpha^2\,\frac{c_1}{m}\,\eta_1^2\,\frac{\partial^2}{\partial\eta_2\delta\eta_1}+\alpha^2\eta_1^2c_0\,\frac{\partial^2}{\partial\eta_2^2}\right.\\ &\left.\left.-\alpha^2\,\frac{\eta_1\eta_2}{m}\,c_1\,\frac{\partial^2}{\partial\eta_2^2}\right\}\right]\langle\phi\rangle,\end{aligned} \tag{32}$$

where c_0 and c_1 are expressed in terms of correlation functions as follows:

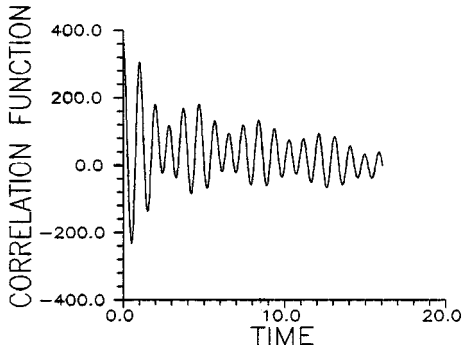

FIG. 1. Plot of the correlation function $\langle\langle \xi(t)\xi(t-\tau)\rangle\rangle$ vs time. Both units are arbitrary.

$$c_0=\int_0^{\alpha}\langle\langle \xi(t)\xi(t-\tau)\rangle\rangle d\tau,$$
$$c_1=\int_0^{\alpha}\langle\langle \xi(t)\xi(t-\tau)\rangle\rangle \tau\, d\tau, \tag{33}$$

Also note that $c=\langle \xi(t)\rangle$.

Note that Eq. (32) is a Fokker-Planck equation which takes into account the quantum fluctuations to a leading order, and that the drift and diffusion coefficients have been derived using the classical dynamical properties of the chaotic system.

On inspection of Eq. (32), the average drift of quantum fluctuations of the dynamical variables η_1 and η_2 corresponding to x and p, respectively, can be written immediately as follows:

$$\overline{\eta_1(t)}\sim\left(\frac{1}{2}\,\overline{\eta_1(0)}+\frac{1}{2mk}\,\overline{\eta_2(0)}\right)\exp(kt) \tag{34}$$

and

$$\overline{\eta_2(t)}\sim\left(\frac{m}{2}\,k\,\overline{\eta_1(0)}+\frac{1}{2}\,\overline{\eta_2(0)}\right)\exp(kt),$$

as t is large.

The average growth of quantum fluctuations is thus exponential in nature, which is in agreement with earlier numerical studies [5–7], and the rate constant of growth k is due to the classical fluctuations of the curvature of the potential as embedded in the correlation function in c_1 in

$$k=\left[\frac{1}{m}\left(2b+\alpha mc+\frac{\alpha^2}{m}\,c_1\right)\right]^{1/2}. \tag{35}$$

B. Numerical verifications

To verify the exponential growth of quantum fluctuations quantitatively, we first consider the classical motion corresponding to the Hamiltonian (25). We choose the parameter values [11] $m=1$, $a=0.5$, $g=10$, and $\omega_0=6.07$, where g includes the effect of the coupling and the driving field amplitude. b is varied from set to set to achieve well developed global chaos for the initial conditions $x_0=-3.5$ and $p_0=0.0$. To calculate classical ensemble average quantities, the averaging is carried over a long time series for the given initial condition. A representative decay of correlation function $\langle\langle \xi(t)\xi(t-\tau)\rangle\rangle$ is shown ($b=10$) in Fig. 1.

TABLE I. Comparison of the rate of divergence of quantum uncertainty calculated numerically (from fully quantum considerations), $k_{\text{numerical}}$, with the rate of divergence calculated theoretically, $k_{\text{theoretical}}$, [from Eq. (35), classical expression].

b	$k_{\text{numerical}}$	$k_{\text{theoretical}}$
12.0	2.86	2.9
10.0	3.0	2.94
8.0	1.97	2.06
5.0	2.62	2.68

It has been pointed out earlier that we take care of fluctuations upto the order of α^2. Since $\alpha\tau_c$ is small, as implied in the theory, it has been possible to subdivide the time axis in the intervals of Δt such that $\Delta t\gg\tau_c$ and also $\alpha\Delta t\ll 1$. That is, η_1 and η_2 do not vary much during a time Δt in which $\xi(t)$ has forgotten its past. Thus, on the coarse-grained level determined by Δt, the process is approximately Markovian. τ_c is thus very short (not zero), and is shorter compared to relevant time scales of the system, i.e., $1/\omega$ and $1/g$. To implement this numerically we consider the first fall of the correlation function, which is fitted by an exponential function of the type $\langle\langle \xi^2\rangle\rangle\exp(-\beta\tau)$ to extract the near-Markovian part of the decay, β being determined by the fit, and calculate the integral c_1 which is expressed as

$$c_1=\int_0^{\alpha}\tau\langle\langle \xi(t)\xi(t-\tau)\rangle\rangle d\tau.$$

Note that $c_1=0$ in the Markovian case.

Having calculated the value of c_1 and also that of $c(=\langle \xi(t)\rangle)$ from a long time series, the classical growth rate constant k in Eq. (35) can be obtained immediately. The entire procedure is repeated for various values of b, as displayed in Table I (second column, $k_{\text{theoretical}}$).

The quantum calculation proceeds by direct quantization of H, with $\hat{x}$ and $\hat{p}$ being position and momentum operators, respectively. We choose harmonic oscillator eigenvectors $\{|n\rangle\}$ as our basis vector defined as

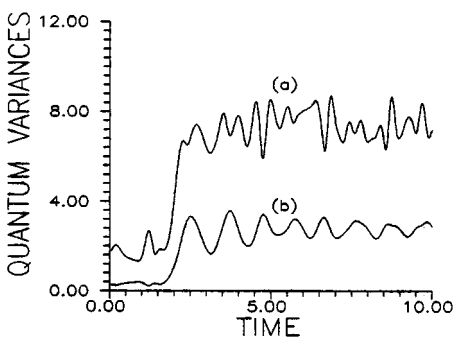

FIG. 2. Plot of quantum variances with time. Curve (a) is a plot of the quantum variance in momentum with time, and curve (b) is a plot of the quantum variance in position with time. Both the units are arbitrary.

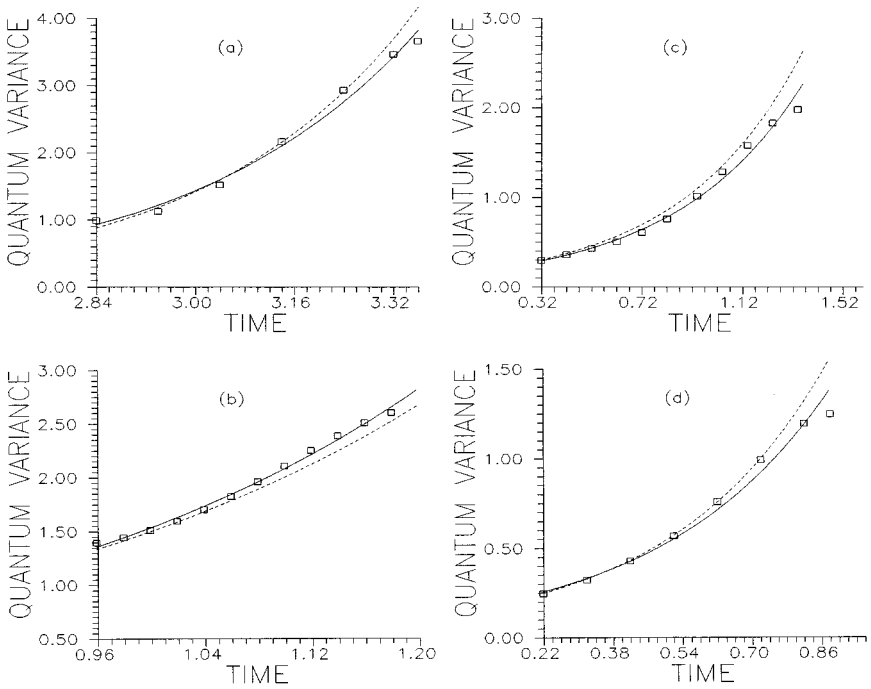

FIG. 3. Plot of the quantum variance in momentum with time for different values of b. The line indicated by the box refers to a fully quantum calculation. The continuous line refers to an exponential fit for the quantum calculation. The dashed line represents the classical solution [Eq. (34)], where k is determined from classical consideration. (a) $b=12.0$, (b) $b=10.0$, (c) $b=8.0$, (d)/ $b=5.0$. (Both units are arbitrary.)

$$\left[\frac{\hat{p}^2}{2m}+\frac{1}{2}\,m\omega^2\hat{x}^2\right]|n\rangle=(n+\tfrac{1}{2})\hbar\omega|n\rangle. \tag{36}$$

For computational purpose we set $\omega=6.25$ and $\hbar=1$. The undriven double-well potential when diagonalized in this basis (we have chosen 120 in number) gives 19 negative energy eigenvalues. The time evolution of the system ψ under a periodic driving field can be followed by the Schrödinger equation

$$i\hbar\dot{c}_n=\sum_m H_{mn}c_m,$$

where

$$c_m=\langle m|\psi(t)\rangle,$$

and the Hamiltonian matrix H_{mn} is as given in [11].

To bring forth quantum-classical correspondence we construct, as our initial wave function, the minimum uncertainty wave packet of the Gaussian form in both position and momentum representations centered around average position x and average momentum p corresponding to initial conditions for the classical trajectory for a typical value of b and g. Thus

$$|\psi(t=0)\rangle=\sum_n c_n(0)|n\rangle,$$

where

$$c_n(0)=\exp(-\tfrac{1}{2}|\alpha|^2)\alpha^{*n}/\sqrt{n!}$$

and

$$\alpha=\left[x+i\left(\frac{1}{m\omega}\right)p\right]\sqrt{m\omega/2}.$$

In Fig. 2 we plot a typical variation of uncertainty in x (corresponding to η_1) and p (corresponding to η_2) for the wave packet centered at $x=-3.5$ and $p=0.0$, and for $b=10.0$. It is evident that after an initial plateau portion the uncertainties in x and p diverge. To make this part prominent we cut off the plateau regions, and in Fig. 3(a)–3(d) plot the quantum variance in x where the exponential divergence is exhibited for various values of b (the line indicated by boxes). The neglect of plateau regions is in accordance with large t, as implied in solution (34) (but t is not too large so that quantum correlations become strong enough to invalidate the semiclassical approximation). It may be noted that these plateau regions are not apparent in the kicked dynamics [6], but may be inherent in many other observations of flows. The plot of quantum variances for various values of b is then fitted with exponential functions to determine the rate constants of divergence ($k_{\text{numerical}}$ as shown in Table I). This is indicated by the continuous lines in Figs. 3(a)–3(d). For comparison of the rates, calculated classically as well as from quantum-mechanical considerations, we superimpose solutions (34) on Figs. 3(a)–3(d) (dashed lines) with the $k_{\text{theoretical}}$ values indicated in Table I. The accuracy of the agreement between the quantum case with the Gaussian wave packet and the corresponding classical case, as exhibited in Figs. 3(a)–3(d) and in Table I, is thus quite satisfactory. It must be emphasized that the accuracy of the agreement rests primarily on the implementation of the near-Markovian character of the dynamics, the correlation time being the shortest time scale compared to the inverse of the driving frequency ω or coupling constant g. Our numerical experience shows that if one accounts for the decay of correlation function within ω^{-1} or g^{-1}, the agreement remains within 10%. This lends support to the statistical description of the fluctuations in the curvature of the potential, which has been shown to be instrumental in several earlier occasions investigated by us [3,4].

C. Discussions

The exponential growth of quantum uncertainty has proved to be an important manifestation of semiclassical chaos. Although Wigner formalism has been explicitly employed in the earlier two [7,10] analyses, (and also in the present one), it is interesting to note that this manifestation is accounted for by using classical arguments. For example, Fox and Eiston [10] have considered a systematic $1/\hbar$ expansion of the Wigner equation similar to the $1/\Omega$ expansion of the master equation by Van Kampen, where the quantum uncertainty was shown to depend only on the contribution of the classical motion to Wigner equation, quantum dynamics being subtle involved in the sense that the uncertainty product puts a constraint in the initial Wigner density (so that it is not just a δ function). However, in a different analysis Bonci *et al*. [7], have shown that a quantum diffusion generating mechanism also contributes to make the growth of the quantum uncertainty faster than when this mechanism is absent. Rather than emphasizing mechanisms, here we stress a different aspect, namely, the statistical description of the curvature of the potential embedded in the classical contribution to the Wigner equation in terms of the theory of multiplicative noise. An offshoot of this treatment is a quantitative analytical expression for the rate of divergence of the quantum uncertainty. We point out in passing that our result is correct up to second order, but exact in the limit in which the correlation time tends to zero.

V. CONCLUSIONS

Classical chaos is characterized by extreme sensitivity to initial conditions. Thus the chaotic dynamics, although deterministic, is stochastic in nature in the statistical sense. It is therefore expected that statistical mechanical formalisms might be useful [3,4,12–14] in the description of classical chaos. We have seen that the theory of multiplicative noise is a good natural description for this purpose in the treatment of classical fluctuations in the curvature of the potential whose correlation is subsequently shown to be instrumental in determining the rate of divergence of quantum variances. We have numerically verified the basic theoretical propositions, and note that the theory is valid for small but rapid fluctuations with a correlation time which is short but finite. This is necessary for a systematic separation of the time scales involved in the dynamics. We hope that the stochastic treatment, which takes into account the arbitrary correlation time, will be useful for further progress in such issues.

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